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PESIT Bangalore South Campus

Hosur Road (1km before Electronic City), Bengaluru -560 100

Department of Basic Science and Humanities

CONTINUOUS INTERNAL EVALUATION TEST -3

Date : 14/05/18		Marks: **60**
Subject & Code : Engineering Mathematics II, 17MAT21		Section: Common to all branches
Name of faculty : KR,DN,NLS,GKJ,GVR,RR,SV		Time : 8.30 to 10.00 a.m.

Note: Answer FIVE full questions choosing any ONE full question from each part.

		PART 1	
1	**a**	Prove that $\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},(m>0,n>0)$ and show that $\Gamma\left(\dfrac{1}{2}\right)=\sqrt{\pi}$	**12**
2	**a**	Prove that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx \times \int_0^1 \frac{1}{\sqrt{1+x^4}}dx = \frac{\pi}{4\sqrt{2}}$	**12**
		PART 2	
3	**a**	Evaluate $L\left\{2^t+\dfrac{\cos 2t-\cos 3t}{t}+t\sin t\right\}$	**6**
	b	Express $f(t)=\begin{cases}1 & 0<t\le 1\\ t & 1<t\le 2\\ t^2 & t>2\end{cases}$ in terms of Unit step function and hence find its Laplace transform.	**6**
4	**a**	Evaluate $L\left\{te^{-2t}\sin 4t+\dfrac{\sin^2 t}{t}\right\}$	**6**
	b	Express $f(t)=\begin{cases}\cos t & 0<t\le \pi\\ \cos 2t & \pi<t\le 2\pi\\ \cos 3t & t>2\pi\end{cases}$ in terms of Unit step function and hence find its Laplace transform.	**6**
		PART 3	

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5	**a**	Given $f(t)=\begin{cases} E & 0<t\le a/2 \\ -E & a/2<t<a \end{cases}$ where f(t+a) = f(t). Show that $L\{f(t)\}=\frac{E}{s}\tanh\left(\frac{as}{4}\right)$	**12**
6	**a**	A periodic function of period $2\pi/\omega$ is defined by $f(t)=\begin{cases} E\sin\omega t, & 0\le t<\pi/\omega \\ 0, & \pi/\omega\le t<2\pi/\omega \end{cases}$ where E and ω are constants. Show that $L\{f(t)\}=\frac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$	**12**
		PART 4	
7	**a**	Find $L^{-1}\left\{\frac{1}{(s-1)(s^2+1)}\right\}$ using convolution theorem	**6**
	b	Find $L^{-1}\left\{\frac{4s+5}{(s+1)^2(s+2)}+\log\left(1-\frac{a^2}{s^2}\right)\right\}$	**6**
8	**a**	Find $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$ using convolution theorem	**6**
	b	Find $L^{-1}\left\{\frac{7s+4}{4s^2+4s+9}\right\}$	**6**
		PART 5	
9	**a**	Solve the equation by Laplace transform method, $y''+4y'+3y=e^{-t}$ given $y(0)=y'(0)=1$.	**12**
10	**a**	Solve the equation by Laplace transform method, $y''+6y'+9y=12t^2e^{-3t}$ given $y(0)=y'(0)=0$	**12**

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Department of Basic Science and Humanities

CONTINUOUS INTERNAL EVALUATION TEST -3 –SOLUTIONS

Date:14/5/18	Marks:**60**
Subject & Code : Engineering Mathematics II, 17MAT21	Section: Common to all branches
Name of faculty : KR,DN,NLS,GKJ,GVR,RR,SV	Time : 8.30 to 10.00 a.m.

Note: Answer FIVE full questions choosing any ONE full question from each part.

PART 1

1 **a** Prove that $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, (m > 0, n > 0)$ and show that $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$ **12**

Solution:

We have by the definition of beta and gamma functions

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \quad \cos^{2n-1}\theta \quad d\theta$$

$$\Gamma(n) = 2\int_0^{\infty} e^{-x^2} \quad x^{2n-1} \quad dx$$

$$\Gamma(m) = 2\int_0^{\infty} e^{-y^2} \quad y^{2m-1} \quad dy$$

$$\Gamma(m+n) = 2\int_0^{\infty} e^{-r^2} \quad y^{2(m+n)-1} \quad dr$$ ....(4 Mark)

Now $\Gamma(m)\Gamma(n) = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2n-1} y^{2m-1} dxdy$ ....(2 Mark)

Changing into polar coordinates we have

$$\Gamma(m)\Gamma(n) = 4\int_{r=0}^{\infty}\int_{\theta=0}^{\pi/2} e^{-(r^2)} r^{2(m+n)-1} \sin^{2m-1}\theta \cos^{2n-1}\theta \quad dr d\theta$$ ....(1 Mark)

$$= 2\int e^{-r^2} r^{2(m+n)-1} dr \quad 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \quad d\theta$$

$$= \Gamma(m+n) \quad \beta(m,n)$$ ....(1 Mark)

Hence the proof.

Consider

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \ \ \cos^{2n-1}\theta \ \ d\theta$$ ....(1 Mark)

$$\beta(\frac{1}{2},\frac{1}{2}) = 2\int_0^{\pi/2} \sin^{2(1/2)-1}\theta \ \ \cos^{2(1/2)-1}\theta \ \ d\theta$$

$= \pi$ ....(2 Mark)

But $\beta(\frac{1}{2},\frac{1}{2}) = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1/2+1/2)} = \pi$.

Hence

$\Gamma(1/2) = \sqrt{\pi}$ ....(1 Mark)

2	a	Prove that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx \times \int_0^1 \frac{1}{\sqrt{1+x^4}}dx = \frac{\pi}{4\sqrt{2}}$	12

Solution:

Let $I_1 = \int_0^1 \frac{x^2}{\sqrt{1-x^4}}dx$

Put $x^4 = \sin^2\theta$.

Therefore,

$$I_1 = \frac{1}{2}\int_0^{\pi/2} \sin^{1/2}\theta \cos^0 \theta d\theta$$ ....(5 Mark)

$$I_1 = \frac{1}{4}\beta\left(\frac{3}{4},\frac{1}{2}\right)$$

Let $I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

Put $x^4 = \tan^2\theta$.

Therefore,

$$I_2 = \frac{1}{2}\int_0^{\pi/4} \sin^{-1/2}\theta \cos^{-1/2} \theta d\theta$$ ....(5 Mark)

$$I_2 = \frac{1}{4\sqrt{2}}\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

Hence

$$I_1 \times I_2 = \frac{1}{16\sqrt{2}} \frac{\Gamma(3/4)\Gamma(1/2)\Gamma(1/4)\Gamma(1/2)}{\Gamma(5/4)\Gamma(3/4)} = \frac{\pi}{4\sqrt{2}}$$ …………………(2 Mark)

PART 2

3 **a** Evaluate $L\left\{2^t + \frac{\cos 2t - \cos 3t}{t} + t\sin t\right\}$ **6**

Solution:

$L\{2^t\}=L\{e^{\log 2.t}\}=\frac{1}{s-\log 2}$ ....(1 Mark)

$$L\left\{\frac{\cos 2t - \cos 3t}{t}\right\} = \int_s^{\infty}\left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right)ds$$ ....(1 Mark)

$$= \frac{1}{2}\left[\log(s^2+4) - \log(s^2+9)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+4}{s^2+9}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\lim_{s\to\infty}\log\left(\frac{1+\frac{4}{s^2}}{1+\frac{9}{s^2}}\right) - \log\left(\frac{s^2+4}{s^2+9}\right)\right]_s^{\infty}.$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s^2+4}{s^2+9}\right)\right]$$ ....(1 Mark)

$$= \left[\log\sqrt{\frac{s^2+9}{s^2+4}}\right]$$ ....(1 Mark)

$L\{t\sin t\}=\frac{2s}{(s^2+1)^2}$ ....(2 Mark)

	b	Express $f(t)=\begin{cases}1 & 0<t\le 1\\ t & 1<t\le 2\\ t^2 & t>2\end{cases}$ in terms of Unit Step function and hence find its Laplace transforms. Solution: f(t) = f_1(t) + [f_2(t)-f_1(t)]u(t-a)+[f_3(t)-f_2(t)]u(t-b) ....(1 Mark) = 1+ [(t-1)u(t-1)]+[t^2-t]u(t-2) ....(1 Mark) Take Laplace on both sides L{f(t)}= 1/s + L{(t-1)u(t-1)}+L{(t^2-t) u(t-2)} ....(2 Mark) $= \frac{1}{s}+\frac{e^{-s}}{s^2}+e^{-2s}\left\{\frac{2}{s^3}+\frac{3}{s^2}+\frac{2}{s}\right\}$ ....(2 Mark)	**6**
4	**a**	Evaluate $L\left\{te^{-2t}\sin 4t+\frac{\sin^2 t}{t}\right\}$ Solution: $L\{\sin 4t\}=\frac{4}{s^2+16}$ ....(1 Mark) $L\{t\sin 4t\}=(-1)\frac{d}{ds}\left(\frac{4}{s^2+16}\right)$ $=\frac{8s}{(s^2+16)^2}$ ....(1 Mark) $L\{e^{-2t}t\sin 4t\}=\frac{8(s+2)}{[s^2+4s+20]^2}$ ....(1 Mark) $L\left\{\frac{\sin^2 t}{t}\right\}=\frac{1}{2}\int_s^\infty\left(\frac{1}{s}-\frac{s}{s^2+4}\right)ds$ ....(1 Mark) $=\frac{1}{2}\left[\log(s)-\frac{1}{2}\log(s^2+4)\right]_s^\infty$	**6**

$$= \frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]$$ ....(1 Mark)

$$= \frac{1}{2}\left[\log\left(\frac{\sqrt{s^2+4}}{s}\right)\right]$$ ....(1 Mark)

b **6**

Express $f(t) = \begin{cases} \cos t & 0 < t \le \pi \\ \cos 2t & \pi < t \le 2\pi \\ \cos 3t & t > 2\pi \end{cases}$ in terms of Unit step function and hence find its Laplace transform.

<u>Solution:</u>

f(t) = $f_1(t)$ + [$f_2(t)$-$f_1(t)$]u(t-a)+[$f_3(t)$-$f_2(t)$]u(t-b) ....(1 Mark)

= cos t +(cos2t-cost)u(t-π)+(cos3t-cos2t)u(t-2π)......................(1 Mark)

Take Laplace on both sides

L{f(t)}= L{cos t}+L(cos2t-cost)u(t-π)+L(cos3t-cos2t)u(t-2π)................(1 Mark)

$$= \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left\{\frac{s}{s^2+9} - \frac{s}{s^2+4}\right\}$$ ....(3 Mark)

PART 3

5 **a** **12**

Given $f(t) = \begin{cases} E & 0 < t \le a/2 \\ -E & a/2 < t < a \end{cases}$ where f(t+a) = f(a). Show that

$$L\{f(t)\} = \frac{E}{s}\tanh\left(\frac{as}{4}\right)$$

The given function is periodic with period T= a. ....(1 Mark)

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\quad dt \text{ ....(1 Mark)}$$

$$= \frac{1}{1-e^{-sa}}\int_0^a e^{-st} f(t)\quad dt \text{ ....(2 Mark)}$$

$$= \frac{1}{1-e^{-sa}}\left\{\int_0^{a/2} e^{-st}E\quad dt + \int_{a/2}^{a} e^{-st}(-E)\quad dt\right\} \text{ ....(2 Mark)}$$

$$= \frac{E}{1-e^{-as}}\left\{\left[\frac{e^{-st/2}}{-s}\right]_0^{a/2} + \left[\frac{e^{-st}}{s}\right]_{a/2}^{a}\right\} \text{ ....(2 Mark)}$$

$$= \frac{E(1-e^{-as/2})}{s(1-e^{-as})} = \frac{E}{s}\tanh\left(\frac{as}{4}\right) \text{ ....(4Mark)}$$

6 **a** A periodic function of period $2\pi/\omega$ is defined by **12**

$$f(t) = \begin{cases} E\sin\omega t, & 0 \le t < \pi/\omega \\ 0, & \pi/\omega \le t < 2\pi/\omega \end{cases}$$

where E and ω are constants.

Show that $L\{f(t)\} = \dfrac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$

<u>Solution:</u>

The given function is periodic with period T= 2π/w. ....(1 Mark)

$$L\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\quad dt \quad \text{......................(1 Mark)}$$

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{2\pi\omega} e^{-st} f(t)\quad dt \text{ ....(2 Mark)}$$

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s/\omega}}\int_0^{\pi/\omega} e^{-st} E\sin\omega t\quad dt \text{ ....(2 Mark)}$$

$$= \frac{E}{1-e^{-2\pi s/\omega}}\left[\frac{e^{-st}}{(-s)^2+w^2}(-s\sin\omega t-\omega\cos\omega t)\right]_0^{\pi/\omega}$$ ....(2 Mark)

$$= \frac{E\omega(1+e^{-\pi s/\omega})}{(s^2+\omega^2)(1-e^{-2\pi s/w})} = \frac{Ew}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$$ ....(4 Mark)

PART 4

7 **a** Find $L^{-1}\left\{\frac{1}{(s-1)(s^2+1)}\right\}$ using convolution theorem **6**

Solution:

F(s) = $\frac{1}{s-1}$ G(s) = $\frac{1}{s^2+1}$

f(t) = e^t ....(1 Mark) g(t) = sin t ....(1 Mark)

$$L^{-1}\left\{\frac{1}{(s-1)(s^2+1)}\right\} = \int_0^t e^u \sin(t-u)du$$ ....(1 Mark)

$$= \left[\frac{e^u}{1+1}\{\sin(t-u)+\cos(t-u)\}\right]_0^t$$ .....(2 Mark)

$$= \frac{1}{2}\left(e^t-\sin t-\cos t\right)$$ ....(1 Mark)

b Find $L^{-1}\left\{\frac{4s+5}{(s+1)^2(s+2)}+\log\left(1-\frac{a^2}{s^2}\right)\right\}$ **6**

Solution:

$$L^{-1}\left\{\frac{4s+5}{(s+1)^2(s+2)}+\log\left(1-\frac{a^2}{s^2}\right)\right\} = L^{-1}\left\{\frac{4s+5}{(s+1)^2(s+2)}\right\}+L^{-1}\left\{\log\left(1-\frac{a^2}{s^2}\right)\right\}$$

Consider

$$L^{-1}\left\{\frac{4s+5}{(s+1)^2(s+2)}\right\} = L^{-1}\left\{\frac{A}{(s+1)}+\frac{B}{(s+1)^2}+\frac{C}{(s+2)}\right\}$$

		(A= 3, B=1, C= -3) ....(2 Mark) $= 3L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{(s+1)^2}\right\} - 3L^{-1}\left\{\frac{1}{(s+2)}\right\}$ ....(1 Mark) $=3e^{-t} + e^{-t}t - 3e^{-2t}$ $L^{-1}\left\{\log\left(1-\frac{a^2}{s^2}\right)\right\} = L^{-1}\left\{\log\left(\frac{s^2-a^2}{s^2}\right)\right\}$ Let F(s) = $\log(s^2-a^2) - \log s^2$ ....(1 Mark) $-F'(s) = -2s/s^2-a^2 + 2/s$ ....(1 Mark) Taking inverse Laplace on both sides we have t f(t) = -2 coshat+2(1) f(t) = $\frac{2(1-\cosh at)}{t}$ ....(1 Mark)	
8	a	Find $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$ using convolution theorem Solution: F(s) = $\frac{s}{s^2+a^2}$ G(s) = $\frac{s}{s^2+b^2}$ f(t) = cos at ....(1 Mark) g(t) = sin bt......................(1 Mark) $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t \cos au \cos(bt-bu)du$ ....(1 Mark) $= \frac{1}{2}\left[\frac{\sin(au+bt-bu)}{a-b} + \frac{\sin(au-bt+bu)}{a+b}\right]_0^t$ ....(2 Mark) $= \frac{1}{a^2-b^2}(a\sin at - b\sin bt)$ ....(1 Mark)	6
	b	Find $L^{-1}\left\{\frac{7s+4}{4s^2+4s+9}\right\}$ Solution:	6

$$L^{-1}\left\{\frac{7s+4}{4s^2+4s+9}\right\}=\frac{1}{4}L^{-1}\left\{\frac{7(s+1/2)+1/2}{(s+1/2)^2+2}\right\}$$ ....(2 Mark)

$$=\frac{e^{-t/2}}{4}\left\{7L^{-1}\left[\frac{s}{s^2+(\sqrt{2})^2}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s^2+(\sqrt{2})^2}\right]\right\}$$ ....(2Mark)

$$=\frac{e^{-t/2}}{4}\left\{7\cos(\sqrt{2})t+\frac{1}{2\sqrt{2}}\sin(\sqrt{2}t)\right\}$$ ....(2 Mark)

PART 5

9		Solve the equation by Laplace transform method, $y''+4y'+3y=e^{-t}$ given $y(0)=y'(0)=1$ Solution: $y''+4y'+3y=e^{-t}$ Take Laplace on both sides, L{y"}+4L{y'}+3L{y} = L{e^{-t}}......................(1 Mark) s^2L{y(t)}-sy(0) -y'(0) +4 [s L{y(t) -y(0)] +3L{y(t)}= $\frac{1}{s+1}$ ....(3 Mark) L{y(t)}= $\frac{s^2+6s+6}{(s+1)^2(s+3)}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{C}{(s+3)}$ ....(2 Mark) (A= 7/4, B=1/2, C= -3/4) ....(3 Mark) y(t) = $\frac{7}{4}L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{(s+1)^2}\right\}-\frac{3}{4}L^{-1}\left\{\frac{1}{(s+3)}\right\}$ ....(1 Mark) = $\frac{7}{4}e^{-t}+\frac{1}{2}e^{-t}t-\frac{3}{4}e^{-3t}$ ....(2 Mark)	**12**
10		Solve the equation by Laplace transform method, $y''+6y'+9y=12t^2e^{-3t}$ given $y(0)=y'(0)=0$.	**12**

Solution:

$y'' + 6y' + 9y = 12t^2 e^{-3t}$

Take Laplace on both sides,

L{y''}+6L{y'}+9L{y} = L{12 t^2 e^{-3t}}..................... ....(2 Mark)

s^2L{y(t)}-sy(0) -y'(0) +6 [s L{y(t) -y(0)] +9 L{y(t)}= $\frac{12(2!)}{(s+3)^3}$ ....(4Mark)

L{y(t)}= $\frac{24}{(s+3)^5}$ ....(3Mark)

y(t) = $L^{-1}\left\{\frac{24}{(s+3)^5}\right\}$......................(1 Mark)

= $e^{-3t}t^4$ ……………………….(2 Mark)